Thermal Conduction in a Tangled Magnetic Field

Benjamin D. G. Chandran* and Steven C. Cowley†
Department of Physics & Astronomy, The University of California at Los Angeles, Los Angeles, California 90095
(Received 1 December 1997; revised manuscript received 27 January 1998)

The suppression of thermal conduction by a static stochastic magnetic field is calculated for different ratios of the field scale length to the collisional mean free path. The effects of magnetic trapping are determined through a two-scale analysis and Monte Carlo particle simulations. In galaxy-cluster cooling flows, thermal conductivity is reduced from the Spitzer value by a factor of order 10^2 to 10^3. [S0031-9007(98)05609-9]

PACS numbers: 52.25.Fi, 95.30.Qd, 98.62.En, 98.62.Ra

In many astrophysical plasmas, the electron mean free path λ is vastly larger than the length scale at which the resistive destruction of magnetic field is significant. It is thus possible in some of these plasmas to develop tangled magnetic fields concentrated on a scale l_B comparable to or shorter than λ. Settings in which this can occur include the intracluster medium in clusters of galaxies [1–3] and transient stages of various astrophysical dynamos [4,5]. In this Letter we calculate the reduction of the thermal conductivity κ by a tangled magnetic field and apply our theory to galaxy-cluster cooling flows [1,6,7]. We consider plasmas with $l_B < \lambda$ as well as $l_B > \lambda$. We assume that these plasmas evolve on time scales long compared to the collision time and that the electron distribution is almost Maxwellian. These assumptions are reasonable for cooling flows given the uncertainties regarding intracluster turbulence. For simplicity, we assume that the magnetic field is static. We focus on the interplay between the spatial divergence of neighboring field lines [8–10] and the trapping of electrons between magnetic mirrors [11,12]. For typical intracluster parameters κ is reduced from the Spitzer value by a factor of order 10^2 to 10^3, a reduction that is substantially larger than in the case of no trapping [8,13–16].

When the gyroradius of thermal electrons ρ_e is much smaller than l_B and λ, the magnetic field controls the motions of individual electrons. Electrons move predominantly along field lines, and regions of relatively strong field act as magnetic mirrors, reflecting electrons that approach from regions of weaker field. The conditions $\rho_e \ll l_B$ and $\rho_e \ll \lambda$ are satisfied by many astrophysical plasmas, including plasmas in which the magnetic field is too weak to be hydrodynamically important. In a static field in the absence of collisions, an electron's kinetic energy $E = m_e v^2/2$ and magnetic moment $\mu = E(1 - \xi^2)/B(x)$ are both conserved. Here, ξ is the electron's pitch angle $v_\parallel/v$, where $v_\parallel$ is the component of the electron's velocity v in the direction of the magnetic field. A particle that starts at position x_0 with pitch angle ξ_0 is thus reflected when it reaches a position x, where $B(x)/B(x_0) = 1/(1 - \xi_0^2)$. Equivalently, a particle with given E and μ can never enter a region in which $B(x) > E/\mu$. Importantly, magnetic mirroring does not depend on the absolute strength of the magnetic field, but only on the ratio of field strengths in different regions. The field strength plays the role of a potential energy, and particles can be trapped in magnetic wells. Trapping restricts an electron's range of motion and thus inhibits thermal conduction.

In a tangled magnetic field, the net displacement Δr of an electron that has moved a distance l along a field line is given by $\Delta r \sim (D_B l)^{1/2}$, where the length D_B is called the "magnetic diffusion coefficient" [9]. In this paper we treat the field as having a single scale l_b so that each stationary field line is a random-walk path of step length l_b, giving $D_B \sim l_B$. A single magnetic scale length may be appropriate for cooling flows, since collisionless damping [5,17] in the intracluster medium may limit the range of scales present in the magnetic field. If an electron's motion along a field line is diffusive with one-dimensional diffusion coefficient $D_\parallel$, then $l \sim (D_\parallel t)^{1/2}$. If the electron stays on the same field line, then $\Delta r \sim t^{1/4}$, and there is no spatial diffusion of electrons [9]. However, as was shown by Rechester and Rosenbluth [8], small motions perpendicular to field lines can restore the diffusive behavior of electrons because of the exponential divergence of neighboring field lines. In a tangled field, the separation d between two closely neighboring magnetic field lines increases on average with distance l along either field line according to the equation

$$d(l) \sim d(0)e^{l/L_K}, \tag{1}$$

where L_K is the Kolmogorov-Lyapunov length [9]. In general, L_K depends on the spectrum of the magnetic field [18,19]. However, since we take the field to have only one scale l_B, $L_K \sim l_B$ [10].

Let us consider a particle at point P in Fig. 1 moving towards point Q initially along the curved solid line, which represents a magnetic field line. Each time the electron moves a distance l_B along the field it drifts a distance $\sim\rho_e$ perpendicular to the magnetic field due to field-strength gradients and field-line curvature. For the moment, however, let us assume that it drifts just once a distance ρ_e from its initial field line onto a neighboring field line. Once it starts following this new field line it

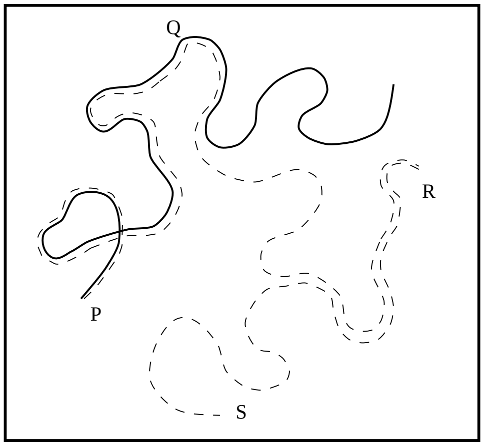

FIG. 1. The curved solid line is a tangled magnetic field line. The dashed lines are electron trajectories.

diverges from its initial field line according to Eq. (1). The electron's perpendicular distance from its initial field line will be $\sim l_B$ after it has moved "the Rechester-Rosenbluth length" L_{RR}, where

$$L_{\mathrm{RR}} \sim l_B \ln(l_B/\rho_e). \tag{2}$$

Because the electron drifts continuously, L_{RR} is a slight overestimate of the distance a particle must travel to reach a perpendicular distance l_B from its initial field line. For simplicity we will ignore this small error. After the electron has moved a distance L_{RR} along the dashed line in Fig. 1 from point P to point R, its subsequent motion is not correlated with its initial field line.

In Fig. 2 we plot an idealized representation of the field strength B as a function of distance l along an electron's trajectory between points P and R. If an electron is trapped in the magnetic well between points P and Q, which are assumed to be separated by a distance less than L_{RR}, then the electron stays close to its initial field line and remains trapped between points P and Q. It is freed only when collisions make E/μ greater than the field strength at P or Q. Perpendicular drifts free the electron only after an extremely long time: in intracluster plasma the time required is far longer than the age of the Universe. On the other hand, if an electron leaving point P passes by point Q and is reflected at point R, then it will not return to point P. Instead of retracing its path, it will drift across field lines and move along the dashed line from point R to point S in Fig. 1. After traveling a distance L_{RR} from point R towards point S, the electron's motion will become uncorrelated from its path between points P and R.

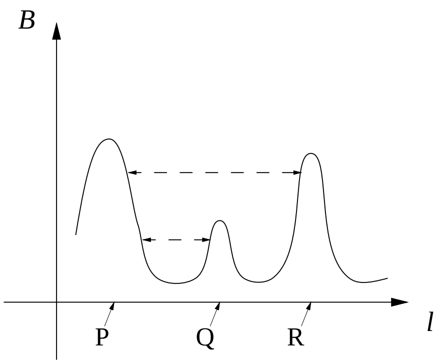

FIG. 2. Simplified plot of the field strength along the dashed line connecting points P and R in Fig. 1. The horizontal dashed lines indicate typical ranges of electron motion between mirror reflections.

We take the thermal conductivity to be approximately equal to the spatial diffusivity of thermal electrons; a more complete theory in which we average over the thermal distribution will be presented in a separate paper. Let us write

$$\kappa = \frac{\Delta^2}{\delta t}, \tag{3}$$

where Δ^2 is the mean-square three-dimensional displacement of a thermal electron during each statistically independent random step, and δt is the duration of each step. Electrons trapped between points P and Q in Fig. 1 take random steps of length l_B along the random magnetic field, but the particles retrace their steps and do not diffuse in space. In order for successive steps to be independent, a single step must be considered as a displacement of at least L_{RR} along a field line. For purposes of numerical estimates, we take the fundamental random step to be $2L_{\mathrm{RR}}$. The corresponding Δ^2 is given by

$$\Delta^2 = \left(\frac{2L_{\mathrm{RR}}}{l_B}\right) l_B^2 = 2L_{\mathrm{RR}} l_B \tag{4}$$

since moving a distance $2L_{\mathrm{RR}}$ along a random field line consists of taking $2L_{\mathrm{RR}}/l_B$ random steps of length l_B. The value of δt depends upon the ratio l_B/λ, where λ is the collisional mean free path. We now turn to an estimate of δt for three different parameter regimes.

We first consider the collisionless limit $(\lambda/L_{\mathrm{RR}}) \to \infty$. Let L_{trap} be the distance an electron travels between mirror reflections. As discussed above, electrons with $L_{\mathrm{trap}} < L_{\mathrm{RR}}$ do not contribute to diffusion in the absence of collisions. On the other hand, electrons with $L_{\mathrm{trap}} \gg L_{\mathrm{RR}}$ stream freely along the magnetic field, traveling a distance $2L_{\mathrm{RR}}$ in a time of roughly $2L_{\mathrm{RR}}/v_T$, where v_T is the electron thermal speed. One can thus estimate the thermal conductivity from Eq. (3) as

$$\kappa = f_{\mathrm{passing}} v_T l_B, \tag{5}$$

where f_{passing} is the fraction of particles with $L_{\mathrm{trap}} > 2L_{\mathrm{RR}}$. We will present a more accurate estimate in more detail in a separate paper, along with calculations of f_{passing} as a function of L_{RR}/l_B for different probability distributions of the random magnetic field. Equation (5) may be relevant to the spatial diffusion of cosmic rays that are only weakly scattered by small-scale magnetic turbulence.

In the semicollisional limit $l_B \ll \lambda \ll 2L_{\mathrm{RR}}$, which is more relevant to cooling flows, the time δt required for an electron to travel a distance $2L_{RR}$ along the field is given by the relation $(2L_{RR})^2 = 2D_\parallel \delta t$. We treat parallel diffusion with the kinetic equation

$$\frac{\partial f}{\partial t} + \xi v_T \frac{\partial f}{\partial l} - \frac{(1-\xi^2)v_T}{2}\frac{d\ln B}{dl}\frac{\partial f}{\partial \xi} = \frac{\nu}{2}\frac{\partial}{\partial \xi}\left[(1-\xi^2)\frac{\partial f}{\partial \xi}\right], \quad (6)$$

where l is distance along a field line and ν is the collision frequency [11,20]. Here we have assumed that collisions change only an electron's pitch angle ξ and that all the electrons move at the thermal speed. An ambipolar electric potential [not included in Eq. (6)] is set up to maintain charge neutrality and prevent the appearance of overwhelming currents. When $l_B \ll f/|\partial f/\partial l|$, this potential has a moderate effect on the heat flux but is too weak to significantly influence single-particle motion or the approximate value of κ given by Eq. (3). We will present calculations including the parallel electric field elsewhere. For constant B, $D_\parallel = v_T\lambda/3$, where $\lambda = v_T/\nu$. For nonconstant B, we write $D_\parallel = \theta v_T \lambda/3$, where θ is the fractional reduction of the parallel diffusivity due to magnetic trapping. Equation (3) can thus be rewritten

$$\kappa = v_T\lambda\theta\frac{l_B}{3L_{RR}}. \quad (7)$$

This is the collisional formula of Rechester and Rosenbluth [8] modified by the mirror-trapping factor θ.

In the magnetized collisional limit $\rho_e \ll \lambda \ll l_B < L_{RR}$ there are no trapped particles, and Eq. (7) with $\theta = 1$ applies. For completeness, we note that in the nonmagnetized collisional limit $\lambda < \rho_e$ electrons do not follow field lines, and κ is not given by Eq. (7) but instead approaches the Spitzer value.

As a first step towards determining θ in Eq. (7), we consider the idealized case in which $B(l) = B_0[1 + \alpha\sin(l/l_B)]$ for $\alpha \in (0,1)$. We solve Eq. (6) with an asymptotic expansion in $\epsilon = l_B/\lambda$ and a two-scale approximation in which $f = f(l,R)$, where $R = l/\epsilon^2$. The distribution function to lowest order in ϵ, denoted f_0, satisfies $\partial f_0/\partial\xi = 0$ and $\partial f_0/\partial l = 0$. Parallel flux is driven by $\partial f_0/\partial R$. To lowest order in ϵ,

$$\theta = \frac{3}{4\langle 1/g\rangle}\int_0^{1/(1+\alpha)} dw \int_w^{1/(1+\alpha)} dz \frac{1}{\langle\sqrt{1-zg}/g\rangle}, \quad (8)$$

where $g \equiv B(l)/B_0$ and the angled brackets denote an average over l. Equation (8) is plotted in Fig. 3. A different result for the reduction of $D_\parallel$ for infinitesimal α has been obtained through a different method by Klepach and Ptuskin [11].

Also shown in Fig. 3 are results from Monte Carlo particle simulations of Eq. (6) for the same periodic magnetic field strength. We treat the Lorentz collision operator with the method of Shanny *et al.* [21]. Each simulation follows 10^4 particles of identical kinetic energy for 20 collision times, and λ is chosen to be $2\pi \times 10^3 l_B$.

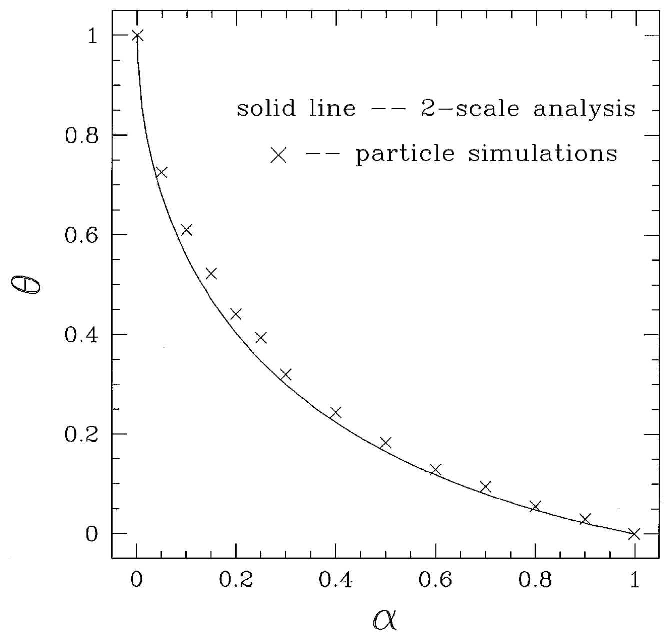

FIG. 3. The reduction factor θ of $D_\parallel$ when $B(l) = B_0[1 + \alpha\sin(l/l_B)]$, where l is distance along a field line, and $l_B \ll \lambda$.

For $B(l) = B_0[1 + \alpha\sin(l/l_B)]$, electrons with $E/\mu < 1 + \alpha$ are trapped in magnetic wells until freed by collisions. These electrons bounce between reflection points on a time scale much shorter than the collision time and to lowest order in ϵ contribute nothing to diffusion along the field. Electrons with $E/\mu > 1 + \alpha$ do contribute to diffusion, and there is a boundary layer at $E/\mu = 1/(1 + \alpha)$. The distribution function $f(l,\mu)$ is continuous across this boundary layer, but the derivative $\partial f/\partial\mu$ undergoes a finite jump. We will describe the boundary-layer structure in detail elsewhere.

When $\alpha = 1 - \delta$ with $\delta \ll 1$, Eq. (8) yields $\theta = 3\delta/16$ to lowest order in δ. The large reduction in $D_\parallel$ when $\alpha \to 1$ reflects the important role of regions of small magnetic field. At any given point, only particles with $\xi > \sqrt{1 - B(l)/B_{max}}$ are able to pass over the regions of maximum field strength B_{max} without reflection. In regions of small B, the width in ξ space of this "passing layer" is small, and collisions more easily scatter particles out of the passing layer. As a result, an electron that is not initially trapped needs to travel only a fraction of its mean free path λ before collisions scatter it onto a trapped trajectory in the l-ξ plane. Even if the electron quickly scatters back onto a passing trajectory, its direction of motion along the field is randomized if it stays in the well for at least one bounce time. Thus, not only does the magnetic well prevent trapped particles from participating in diffusion, but it reduces the parallel diffusivity of passing particles as well.

When the magnetic field is random, an analytic treatment is difficult because for some electrons $L_{trap} < \lambda$ while for others $L_{trap} > \lambda$. We treat parallel diffusion in a random field with the use of Monte Carlo particle

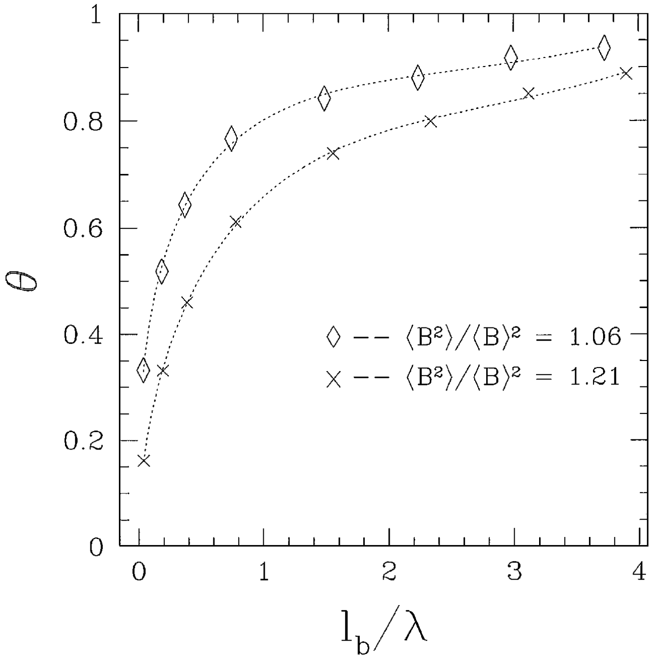

FIG. 4. Numerical results for the reduction factor θ of $D_\parallel$ for two different probability distribution functions of the random magnetic field.

simulations of Eq. (6). We take the field strength $B(l)$ to be given by

$$B(l) = \sum_{i=1}^{i=N} a_i e^{-(l-l_i)^2/2}, \tag{9}$$

where a_i and l_i are random numbers uniformly distributed on the intervals $0 < a_i < 1$ and $-L < l_i < L$. The length of the simulation domain $2L$ is chosen so that electrons do not leave the domain during the simulation. To evaluate Eq. (9) to machine accuracy at any given l, it is sufficient to include in the sum only those terms for which $|l - l_i| < 6$. Increasing N/L changes the probability distribution of $|B|$, reducing the fractional variations in the field strength and the effectiveness of particle trapping. A measure of the fractional variations in $|B|$ is $\langle B^2\rangle/\langle |B|\rangle^2$, where the angled brackets indicate an average over l. It should be emphasized that $\langle |B|\rangle$ is not the strength of the mean field vector. By assuming a single magnetic scale length l_B, we have implicitly taken the mean field vector to be small. The reduction factor θ of $D_\parallel$ is plotted in Fig. 4 as a function of l_B/λ for two different values of N, one of which corresponds to $\langle B^2\rangle/\langle |B|\rangle^2 = 1.06$, the other of which corresponds to $\langle B^2\rangle/\langle |B|\rangle^2 = 1.21$. Numerically, l_B is taken to be $\langle |B|\rangle/\langle |dB/dl|\rangle$. Each simulation follows 4×10^4 particles of identical kinetic energy for 40 collision times. Increasing ν reduces the effects of trapping, thereby increasing θ. However, increasing ν reduces $D_\parallel = \theta v_T^2/3\nu$ since collisions inhibit the flow of particles along the field.

In conclusion, let us consider a numerical example relevant to intracluster plasma. In a 10^{-6} G field, thermal electrons in a 10^8 K plasma have gyroradii of 2.2×10^8 cm. Taking $l_B = 4.0$ kpc, we find from Eq. (2) that $L_{\mathrm{RR}} = 32 l_B$. For $\lambda = 20$ kpc and $\langle B^2\rangle/\langle |B|\rangle^2 = 1.21$, the Monte Carlo particle simulations of Fig. 4 yield $\theta = 0.33$. Equation (7) then implies that $\kappa = v_T\lambda/280$. For a nonmagnetized plasma, Eq. (3) yields the approximate Spitzer thermal conductivity $\kappa_{\mathrm{S}} = v_T\lambda$. For characteristic intracluster parameters, κ is thus on the order of $1/300$ of the Spitzer value. If l_b/λ remains constant as λ decreases towards a cluster's center, then a similar reduction of κ occurs within the cluster's core, consistent with the neglect of thermal conduction in homogeneous cooling-flow models of intracluster plasma [1,6,7].

It is a pleasure to acknowledge several valuable discussions with Rick Sydora regarding the Monte Carlo simulations. This work was supported by NSF Grant No. AST 97-13241 and by the Department of Energy.

*Electronic address: chandran@physics.ucla.edu
†Electronic address: cowley@physics.ucla.edu

[1] A. C. Fabian, Annu. Rev. Astron. Astrophys. **32**, 277 (1994).
[2] E. G. Zweibel and C. Heiles, Nature (London) **385**, 131 (1997).
[3] P. P. Kronberg, Rep. Prog. Phys. **57**, 325 (1994).
[4] R. M. Kulsrud and S. W. Anderson, Astrophys. J. **396**, 606 (1992).
[5] R. M. Kulsrud, S. C. Cowley, A. Gruzinov, and R. Sudan, Phys. Rep. **283**, 213 (1997).
[6] C. L. Sarazin, *X-ray Emission from Clusters of Galaxies* (Cambridge University Press, Cambridge, England, 1988).
[7] J. Binney and L. L. Cowie, Astrophys. J. **247**, 464 (1981).
[8] A. B. Rechester and M. N. Rosenbluth, Phys. Rev. Lett. **40**, 38 (1978).
[9] J. A. Krommes, C. Oberman, and R. G. Kleva, J. Plasma Phys. **30**, 11 (1983).
[10] L. G. Chuvilgin and V. S. Ptuskin, Astron. Astrophys. **279**, 278 (1993).
[11] E. G. Klepach and V. S. Ptuskin, Astron. Lett. **21**, 411 (1995).
[12] K. J. Borkowski, S. A. Balbus, and C. C. Fristrom, Astrophys. J. **355**, 501 (1990).
[13] R. Rosner and W. H. Tucker, Astrophys. J. **338**, 761 (1989).
[14] P. C. Tribble, Mon. Not. R. Astron. Soc. **238**, 1247 (1989).
[15] L. Tao, Mon. Not. R. Astron. Soc. **275**, 965 (1995).
[16] S. Pistinner and G. Shaviv, Astrophys. J. **459**, 147 (1996).
[17] A. Barnes, Phys. Fluids **9**, 1483 (1966).
[18] J. Skilling, I. McIvor, and J. A. Holmes, Mon. Not. R. Astron. Soc. **167**, 87P (1974).
[19] J. R. Jokipii, Astron. J. **183**, 1029 (1973).
[20] F. L. Hinton and R. D. Hazeltine, Rev. Mod. Phys. **48**, 239 (1976)
[21] R. Shanny, J. M. Dawson, and J. M. Greene, Phys. Fluids **10**, 1281 (1967).